

Almost sure dimension of projections along chains

Laurent Dufloux

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Abstract

We prove a version of Marstrand's projection Theorem with respect to the foliation of the three dimensional sphere S^3 by chains.¹

1 Introduction

Before introducing our results, let us recall Marstrand's classical projection Theorem.

Marstrand's original Theorem [8] deals with Euclidean spaces; if μ is some finite Borel measure on \mathbf{R}^d ($d \geq 2$) and m is an integer, $1 \leq m \leq d - 1$, then for almost every m -plane V in \mathbf{R}^d , the Hausdorff dimension of the (orthogonal) projection of μ onto V is "as high as it can be". See [9] for details.

Marstrand's Theorem can be rephrased in the language of foliations. To any $(d - m)$ -plane $W \in \mathcal{G}(d, d - m)$ we may associate the foliation of \mathbf{R}^d whose leaves are the translates of W ; this foliation admits a natural transversal which is isometric to \mathbf{R}^m . The orthogonal projection onto the orthogonal of W is just the quotient mapping along this foliation.

In this paper, we look at the foliation of S^3 with chains. We note that all our results make sense and remain true in the $(2d + 1)$ -sphere S^{2d+1} ; we stick to dimension 3 for simplicity. What is a chain? If S^3 is identified with the subset of the complex projective plane

$$\{[1 : x_1 : x_2] \in \mathbf{P}_{\mathbf{C}}^2 ; 1 = |x_1|^2 + |x_2|^2\}$$

then chains are non-empty intersections of S^3 with complex projective lines.

Through two points $x, y \in S^3$ there lies one and only one chain L . We can thus consider the foliation of $S^3 \setminus \{x\}$ whose leaves are the chains passing through x . The space of such foliations is endowed with a natural measure, that is, the usual Lebesgue measure on S^3 .

Informally, we prove the following

Theorem. *Endow S^3 with the usual Euclidean metric. Then the foliation of S^3 with chains satisfies the conclusion of Marstrand's Theorem.*

In other words, if μ is some Borel measure on S^3 , then for (Lebesgue) almost every $x \in S^3$, the Hausdorff dimension of the projection of μ

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along chains passing through x is “as high as it can be” (relatively to the dimension of μ with respect to the Euclidean metric). See Section 3.

Even though our results are Euclidean in nature, our motivation comes from sub-Riemannian geometry – more precisely, the Gromov problem for limit sets of discrete groups of complex hyperbolic isometries.

The boundary $\partial\mathbf{H}_{\mathbb{C}}^2$ of the complex hyperbolic plane identifies with S^3 endowed with a different metric, namely, the Gromov metric (see (1) for the definition). The Hausdorff dimension of $\partial\mathbf{H}_{\mathbb{C}}^2$ with respect to the Gromov metric is 4, whereas its dimension with respect to the Euclidean metric is, of course, 3.

Let A be some Borel subset of $\partial\mathbf{H}_{\mathbb{C}}^2$.

Question. *Let α , resp. β , be the Hausdorff dimension of A with respect to the Euclidean metric, resp. the Gromov metric.*

How are α and β related?

This is a particular instance of Gromov’s dimension comparison problem, [6]. In general, when A has no additional structure, this problem was solved by Balogh et al. in [3]. All that can be said is that

$$\sup \left\{ \frac{\beta}{2}, \beta - 1 \right\} \leq \alpha \leq \inf \left\{ \beta, 1 + \frac{\beta}{2} \right\}$$

and these inequalities are sharp.

The difference between Gromov and Euclidean metrics can be put in the following informal way: transversally to chains, they are comparable, whereas along chains, the Gromov metric is comparable to the square root of the Euclidean metric. See Lemmas 4 and 5. If in some way A has “too much” dimension along chains, then there will be a “jump” in dimension when passing from Euclidean metric to Gromov metric.

The Marstrand-type results we prove in this paper imply in particular that the dimension of a given Borel set A along a random chain is “as small as it can be”. Bear in mind that S^3 is always endowed with the Euclidean metric.

Our motivation comes from the case when A is the limit set of a Zariski-dense discrete group of complex hyperbolic isometries which is also assumed, for example, to be convex-cocompact. In [4] I conjecture that for such a limit set, the dimension along chains should be as small as possible and I deduce a (conjectural) formula for the dimension of the limit set with respect to the Euclidean metric.

I shall not provide more details here. To illustrate the relevance of our Marstrand-type results, let us simply mention that Proposition 6 below allows for an easy proof that, in the setting of [4],

$$\underline{\dim}^T(\text{BMS}, N/Z) \geq \frac{1}{2} \dim(\text{BMS}, Z)$$

which implies in particular that the Hausdorff dimension of the limit set Λ_{Γ} with respect to the Euclidean metric satisfies the lower bound

$$\dim_{\text{H}}(\Lambda_{\Gamma}) \geq \frac{2}{3} \delta_{\Gamma}.$$

When $\delta_{\Gamma} < 3$ this improves on the strict inequality

$$\dim_{\text{H}}(\Lambda_{\Gamma}) > \sup \left\{ \delta_{\Gamma} - 1, \frac{\delta_{\Gamma}}{2} \right\}$$

(which is Proposition 38 in [4]). I hope to come back to this problem in a later paper.

Here are two other examples of sets with special structures for which the Gromov dimension comparison problem yields exact formulas:

- horizontal fractals in the Heisenberg group [2],
- random cut-out sets in the Heisenberg group (joint work with Ville Suomala, to be published).

Before ending this introduction, let us briefly recall the relation between $\partial\mathbf{H}_{\mathbb{C}}^2$ and the Heisenberg group. If we send one point $x \in \partial\mathbf{H}_{\mathbb{C}}^2$ at infinity, the complement $\partial\mathbf{H}_{\mathbb{C}}^2 \setminus \{x\}$ identifies in a natural way with the Heisenberg group \mathbf{H} ; the translates of the center Z of \mathbf{H} are the images of the chains passing through x . In Goldman's terminology ([5]), these are "infinite chains". The image in \mathbf{H} of a chain which does not pass through x is a "finite chain".

We will not emphasize Heisenberg group viewpoint in this paper, because dealing with $\partial\mathbf{H}_{\mathbb{C}}^2$ and S^3 inside $\mathbf{P}_{\mathbb{C}}^2$ is computationally much easier. The reader who is more interested in Heisenberg group can easily translate our results into that setting. Other Marstrand-type Theorems have been proved in the Heisenberg group *endowed with the Heisenberg metric* by Balogh *et al.* in [1]. The foliations considered in this work are different from the foliation by chains we are interested in.

The plan of the paper is as follows. In section 2 we introduce the "ball model" of the complex hyperbolic plane and some facts about chains and the metric estimates we need. Our approach is very "linear algebraic" and makes heavy use of the duality between chains and "negative vectors". This allows for an easy proof of the main formula (Theorem 1); this formula says, essentially, that the chain foliation is "transversal" in the sense of Peres and Schlag, see [11]; we also show that the chain foliation along a fixed chain is transversal as well (subsection 3.2). In section 3 we state the analogues of Marstrand's classical Theorems, all the results have standard proof using Theorem 1 and we dispense ourselves with writing down the well-known arguments.

Some words about notations. If f and g are some functions such that

$$\frac{1}{C}f \leq g \leq Cf$$

for some constant $C > 1$, we write $f \asymp g$ and if we want to say that C depends on some data X , we write $f \asymp_X g$. Likewise, if

$$f \leq Cg$$

we write $f \lesssim g$ and if C depends on the data X we write $f \lesssim_X g$.

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2 The complex hyperbolic plane in the complex projective plane

2.1 Definition

Let q be the usual Hermitian form of signature $(1, 2)$ on \mathbf{C}^3 ,

$$q(u) = |u_0|^2 - |u_1|^2 - |u_2|^2$$

and let $\langle \cdot, \cdot \rangle$ be the associated polar form,

$$\langle u, v \rangle = \overline{u_0}v_0 - \overline{u_1}v_1 - \overline{u_2}v_2.$$

We also endow \mathbf{C}^3 with the usual Euclidean norm $\|\cdot\|$ as well as the inner scalar product

$$u \cdot v = \overline{u_0}v_0 + \overline{u_1}v_1 + \overline{u_2}v_2$$

so that

$$\|u\|^2 = u \cdot u.$$

If u is some non-zero vector of \mathbf{C}^3 , u^\perp will always be the complex plane which is orthogonal to u with respect to $\langle \cdot, \cdot \rangle$.

Remark 1. *The fact that the canonical basis (e_0, e_1, e_2) is orthogonal for both $\langle \cdot, \cdot \rangle$ and the inner scalar product, and that $|q(e_i)| = 1$ for $i = 1, 2, 3$, will play a role in obtaining exact formulas in this section.*

We now define the complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^2$. This is the image of the open set

$$\{u \in \mathbf{C}^3 \setminus \{0\} ; q > 0\}$$

through the quotient mapping

$$\begin{array}{ccc} \mathbf{C}^3 \setminus \{0\} & \rightarrow & \mathbf{P}_{\mathbf{C}}^2 \\ u & \mapsto & [u] \end{array}.$$

The usual hyperbolic metric on $\mathbf{H}_{\mathbf{C}}^2$ is defined in the following way. For any $u \in \mathbf{C}^3$ such that $q(u) > 0$, the restriction of q to u^\perp is negative definite (by virtue of Sylvester's law of inertia); we let g_u be the restriction to u^\perp of

$$-\frac{1}{q(u)}q.$$

Then the push-forward of g through the quotient mapping

$$\{q \neq 0\} \rightarrow \mathbf{H}_{\mathbf{C}}^2$$

is the Hermitian metric (on the tangent bundle of $\mathbf{H}_{\mathbf{C}}^2$) which gives rise to the hyperbolic metric on $\mathbf{H}_{\mathbf{C}}^2$.

The boundary $\partial\mathbf{H}_{\mathbf{C}}^2$ of $\mathbf{H}_{\mathbf{C}}^2$ is the set of all $[u] \in \mathbf{P}_{\mathbf{C}}^2$ such that $q(u) = 0$, *i.e.* (in homogeneous coordinates)

$$\partial\mathbf{H}_{\mathbf{C}}^2 = \{[1 : u_1 : u_2] ; |u_1|^2 + |u_2|^2 = 1\}$$

whereas

$$\mathbf{H}_{\mathbf{C}}^2 = \{[1 : u_1 : u_2] ; |u_1|^2 + |u_2|^2 < 1\}$$

and we can see that $\mathbf{H}_{\mathbf{C}}^2$ (resp. $\partial\mathbf{H}_{\mathbf{C}}^2$) identifies with the 4-ball \mathbf{B}^4 (resp. with the 3-sphere \mathbf{S}^3).

The complex hyperbolic plane is a classic object of study but in this paper we are only going to be concerned with the boundary. Let us introduce the Gromov metric on the boundary:

$$d_G([u], [v]) = \sqrt{\frac{|\langle u, v \rangle|}{\|u\| \|v\|}} \quad (1)$$

where $[u], [v] \in \partial\mathbf{H}_{\mathbf{C}}^2$.

The Gromov metric will not be essential in this paper, where we deal mostly with the Euclidean metric on the boundary, to be introduced in the following subsection.

2.2 The standard metric in projective space

2.2.1 Definition

Recall that \mathbf{C}^3 is endowed with an inner scalar product $(u, v) \mapsto u \cdot v$. Consider the second exterior power $\bigwedge^2 \mathbf{C}^3$. It is canonically endowed with an inner scalar product defined by

$$(u_1 \wedge v_1) \cdot (u_2 \wedge v_2) = \begin{vmatrix} u_1 \cdot u_2 & u_1 \cdot v_2 \\ v_1 \cdot u_2 & v_1 \cdot v_2 \end{vmatrix}$$

(see Bourbaki, Algebra, IX, §1.9). In the same way, the third exterior power $\bigwedge^3 \mathbf{C}^3$ (which is one-dimensional) is endowed with the inner scalar product defined by

$$(u_1 \wedge v_1 \wedge w_1) \cdot (u_2 \wedge v_2 \wedge w_2) = \begin{vmatrix} u_1 \cdot u_2 & u_1 \cdot v_2 & u_1 \cdot w_2 \\ v_1 \cdot u_2 & v_1 \cdot v_2 & v_1 \cdot w_2 \\ w_1 \cdot u_2 & w_1 \cdot v_2 & w_1 \cdot w_2 \end{vmatrix}.$$

We denote by $\|u \wedge v\|$ the associated Euclidean norm on $\bigwedge^2 \mathbf{C}^3$.

Definition 1. For any $[u], [v] \in \mathbf{P}_{\mathbf{C}}^2$, the Euclidean distance from $[u]$ to $[v]$ is

$$d([u], [v]) = \frac{\|u \wedge v\|}{\|u\|\|v\|}.$$

The restriction of d to $\partial \mathbf{H}_{\mathbf{C}}^2$ will be called the Euclidean metric on the boundary.

2.2.2 Distance from a point to a projective line

Let L be a complex projective line in $\mathbf{P}_{\mathbf{C}}^2$ and choose some linear form $\alpha : \mathbf{C}^3 \rightarrow \mathbf{C}$ such that $L = \mathbf{P}(\text{Ker } \alpha)$.

For any $[u] \in \mathbf{P}_{\mathbf{C}}^2$, the distance (with respect to the standard metric) from $[u]$ to L is given, as the reader may check, by duality the formula

$$d([u], L) = \frac{|\alpha(u)|}{\|u\|\|\alpha\|} \quad (2)$$

where $\|\alpha\|$ is the usual operator norm

$$\|\alpha\| = \sup_{v \neq 0} \frac{|\alpha(v)|}{\|v\|}.$$

In this paper we will need a formula giving the distance from a point $[u]$ to the projective line passing through two points $[v], [w]$. This is an elaboration of the previous formula.

Let us state some general facts. The canonical basis (e_0, e_1, e_2) of \mathbf{C}^3 yields an isomorphism

$$\theta : \bigwedge^2 \mathbf{C}^3 \rightarrow (\mathbf{C}^3)^*$$

(where $(\mathbf{C}^3)^*$ is the dual space of \mathbf{C}^3); namely, for any $u \wedge v \in \bigwedge^2 \mathbf{C}^3$, $\theta(u \wedge v)$ is the linear form characterized by the following equality

$$(\theta(u \wedge v)(w))e_0 \wedge e_1 \wedge e_2 = u \wedge v \wedge w$$

for any $w \in \mathbf{C}^3$.

We reformulate the previous fact: for any $u \wedge v \in \bigwedge^2 \mathbf{C}^3$, the mapping

$$\begin{array}{ccc} \mathbf{C}^3 & \rightarrow & \bigwedge^3 \mathbf{C}^3 \\ w & \mapsto & u \wedge v \wedge w \end{array}$$

is an isomorphism, and $\bigwedge^3 \mathbf{C}^3$ is isomorphic to \mathbf{C} via the mapping

$$\begin{aligned} \mathbf{C} &\rightarrow \bigwedge^3 \mathbf{C}^3 \\ \lambda &\mapsto \lambda e_0 \wedge e_1 \wedge e_2 \end{aligned}$$

Of course if $[v], [w]$ are distinct points in $\mathbf{P}_{\mathbf{C}}^2$, the kernel of $\theta(v \wedge w)$ is the complex plane passing through v and w , so that equation (2) yields, for any $[u] \in \mathbf{P}_{\mathbf{C}}^2$,

$$d([u], \mathbf{P}(\mathbf{C}v \oplus \mathbf{C}w)) = \frac{|\theta(v \wedge w)(u)|}{\|\theta(v \wedge w)\| \|u\|}. \quad (3)$$

Lemma 1. *If $[v], [w]$ are distinct points of $\mathbf{P}_{\mathbf{C}}^2$, then for any $[u] \in \mathbf{P}_{\mathbf{C}}^2$,*

$$d([u], \mathbf{P}(\mathbf{C}v \oplus \mathbf{C}w)) = \frac{\|u \wedge v \wedge w\|}{\|u\| \|v \wedge w\|}. \quad (4)$$

In this formula, $\|u \wedge v \wedge w\|$ is the norm associated to the inner scalar product on $\bigwedge^3 \mathbf{C}^3$ defined in the previous subsection; of course, $\|e_0 \wedge e_1 \wedge e_2\| = 1$ so that $|\theta(v \wedge w)(u)| = \|u \wedge v \wedge w\|$. Also, it is easy to check that θ is an isometry (see remark 1), hence $\|\theta(v \wedge w)\| = \|v \wedge w\|$ and equation (4) now follows from (3).

2.2.3 Distance from a point to a chain on the boundary

Lemma 2. *Let K be a compact subset of $\mathbf{P}_{\mathbf{C}}^2$ which does not meet $\overline{\mathbf{H}_{\mathbf{C}}^2} = \mathbf{H}_{\mathbf{C}}^2 \cup \partial \mathbf{H}_{\mathbf{C}}^2$. For any $x \in \partial \mathbf{H}_{\mathbf{C}}^2$,*

$$d(x, \mathbf{P}(w^\perp) \cap \partial \mathbf{H}_{\mathbf{C}}^2) \asymp_K d(x, w^\perp)$$

for any $w \in K$.

To understand what is going on in this Lemma, look at a sequence w_n of points of $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$ which converges towards a point $w \in \partial \mathbf{H}_{\mathbf{C}}^2$; the sequence of chains $w_n^\perp \cap \partial \mathbf{H}_{\mathbf{C}}^2$ then converges (in the usual Hausdorff topology) towards the single point w .

On the other hand, if $w \in \partial \mathbf{H}_{\mathbf{C}}^2$, it follows from Formula (2) that

$$d(x, \mathbf{P}(w^\perp)) = \frac{|\langle u, w \rangle|}{\|u\| \|w\|}$$

(where $x = [u]$); the right-hand side is by definition the square of the Gromov distance $d_G(x, [w])$, i.e.

$$d(x, \mathbf{P}(w^\perp)) = d_G(x, [w])^2.$$

In general, $d_G(x, [w])^2$ does not behave like $d(x, [w])$.

This shows that when a chain $L \cap \partial \mathbf{H}_{\mathbf{C}}^2$ (where L is a projective line that meets $\mathbf{H}_{\mathbf{C}}^2$) is very small, almost equal to a single point $[w] \in \partial \mathbf{H}_{\mathbf{C}}^2$, the distance $d(x, L)$ behaves – when it is not too small – like $d_G(x, w)^2$, not like $d(x, w)$.

We now prove the Lemma.

Proof. If $w \in \mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$ is fixed, it is easy to check that

$$d(x, \mathbf{P}(w^\perp) \cap \partial \mathbf{H}_{\mathbf{C}}^2) \asymp_w d(x, w^\perp)$$

(take for example $w = [0 : 0 : 1]$).

Now let K be some compact subset of $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$ and fix some $w \in K$. Let G be the group $\mathbf{PU}(1, 2)$ of isometries of $\mathbf{H}_{\mathbf{C}}^2$. By virtue of Witt's transitivity Theorem, G acts transitively on $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$. The subset $G_K = \{g \in G ; gw \in K\}$ is compact. In particular, if g belongs to G_K , the singular values of g and $\bigwedge^2 g$ belong to some compact subset of $]0, \infty[$.

We deduce that for any $x \in \partial\mathbf{H}_{\mathbf{C}}^2$, and any $g \in G_K$,

$$d(x, g\mathbf{P}(w^\perp)) \asymp_K d(g^{-1}x, \mathbf{P}(w^\perp))$$

and likewise

$$d(x, g\mathbf{P}(w^\perp) \cap \partial\mathbf{H}_{\mathbf{C}}^2) \asymp_K d(g^{-1}x, \mathbf{P}(w^\perp) \cap \partial\mathbf{H}_{\mathbf{C}}^2)$$

where all the constants implied depend only on K ; then, as we remarked in the beginning of the argument, for any $y \in \partial\mathbf{H}_{\mathbf{C}}^2$,

$$d(y, \mathbf{P}(w^\perp)) \asymp_w d(y, \mathbf{P}(w^\perp) \cap \partial\mathbf{H}_{\mathbf{C}}^2)$$

hence the conclusion by letting $y = g^{-1}x$. □

2.3 The Cauchy-Riemann projection

2.3.1 Definition

Let x be some point of $\partial\mathbf{H}_{\mathbf{C}}^2$. The projective line $\mathbf{P}(x^\perp)$ is tangent to $\partial\mathbf{H}_{\mathbf{C}}^2$ at x . For any $p \in \partial\mathbf{H}_{\mathbf{C}}^2$ other than x , the projective lines $\mathbf{P}(x^\perp)$ and $\mathbf{P}(p^\perp)$ are distinct and thus their intersection is a single point of $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$. This defines a mapping

$$\begin{array}{ccc} \pi_x : \partial\mathbf{H}_{\mathbf{C}}^2 \setminus \{x\} & \rightarrow & \mathbf{P}(x^\perp) \setminus \{x\} \\ p & \mapsto & \mathbf{P}(x^\perp) \cap \mathbf{P}(p^\perp) \end{array} \quad (5)$$

Our main interest is in understanding the distance

$$d(\pi_x(p), \pi_x(q)).$$

Once again we will prove, by way of elementary linear algebra, a formula involving exterior products.

We introduce an isomorphism

$$\kappa : \bigwedge^2 \mathbf{C}^3 \rightarrow \mathbf{C}^3$$

in the following way. For any $u \wedge v \in \bigwedge^2 \mathbf{C}^3$, $\kappa(u \wedge v)$ is the unique element of \mathbf{C}^3 that satisfies

$$\langle \kappa(u \wedge v), w \rangle e_0 \wedge e_1 \wedge e_2 = u \wedge v \wedge w.$$

In particular, $\kappa(u \wedge v)^\perp = \mathbf{C}u \oplus \mathbf{C}v$ if $u \wedge v \neq 0$. By definition, we then have

$$\pi_x(p) = [\kappa(u \wedge v)]$$

for any distinct $x = [u]$ and $p = [v]$ in $\partial\mathbf{H}_{\mathbf{C}}^2$. Once again, it is easy to check that κ is an isometry. Now consider the second exterior product of $\bigwedge^2 \mathbf{C}^3$,

$$\bigwedge^2 (\bigwedge^2 \mathbf{C}^3).$$

Pay attention to the fact that $(u_1 \wedge v_1) \wedge (u_2 \wedge v_2)$ is an element of $\bigwedge^2(\bigwedge^2 \mathbf{C}^3)$ and that we cannot remove the parentheses; indeed $\bigwedge^4 \mathbf{C}^3$ is trivial and $u_1 \wedge v_1 \wedge u_2 \wedge v_2 = 0$.

Since $\bigwedge^2 \mathbf{C}^3$ is endowed with an inner scalar product, so is $\bigwedge^2(\bigwedge^2 \mathbf{C}^3)$. The mapping

$$\bigwedge^2 \kappa : \bigwedge^2(\bigwedge^2 \mathbf{C}^3) \rightarrow \bigwedge^2 \mathbf{C}^3$$

defined by

$$\bigwedge^2 \kappa((u_1 \wedge v_1) \wedge (u_2 \wedge v_2)) = \kappa(u_1 \wedge v_1) \wedge \kappa(u_2 \wedge v_2)$$

is still an isometry. For any $u, v, w \in \mathbf{C}^3$, we get

$$\|\kappa(u \wedge v) \wedge \kappa(u \wedge w)\| = \|(u \wedge v) \wedge (u \wedge w)\|.$$

We have thus proved the following

Lemma 3. *For any $x \in \partial \mathbf{H}_{\mathbf{C}}^2$ and any $p, q \in \partial \mathbf{H}_{\mathbf{C}}^2 \setminus \{x\}$,*

$$d(\pi_x(p), \pi_x(q)) = \frac{\|(u \wedge v) \wedge (u \wedge w)\|}{\|u \wedge v\| \|u \wedge w\|} \quad (6)$$

where $x = [u]$, $p = [v]$ and $q = [w]$.

The following Lemmas will not be needed in this paper, we state them because it is easy at this point to clarify the relation between the Euclidean and Gromov metrics.

Lemma 4. *Let K be a compact subset of $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$. Then for any $[w] \in K$ and any x, y on the chain $\mathbf{P}(w^\perp) \cap \partial \mathbf{H}_{\mathbf{C}}^2$,*

$$d_G(x, y) \asymp_K d(x, y)^{1/2}. \quad (7)$$

This gives a precise meaning to the fact that *along chains, the Gromov metric is the square of the Euclidean metric*; the constant explodes as the chain “degenerates” into a point.

Proof. Let $x = [u]$ and $y = [v]$; we may assume that $x \neq y$. All we have to do is show that

$$\frac{|\langle u, v \rangle|}{\|u \wedge v\|} \asymp_K 1.$$

It is easy to check that the left-hand side depends only on $u \wedge v$ and is a continuous function of $[\kappa(u \wedge v)] \in K$. Hence the Lemma. \square

Transversally to chains, the Euclidean and Gromov metrics are comparable:

Lemma 5. *Let K be a compact subset of $\mathbf{P}_{\mathbf{C}}^2 \setminus \overline{\mathbf{H}_{\mathbf{C}}^2}$. For any chain $L = \mathbf{P}(w^\perp) \cap \partial \mathbf{H}_{\mathbf{C}}^2$ where $[w] \in K$, and any $x \in \partial \mathbf{H}_{\mathbf{C}}^2$,*

$$d_G(x, L) \asymp_K d(x, L).$$

We skip the easy proof.

2.3.2 Main formula

We are now ready to prove the following

Theorem 1. *Let $x \in \partial\mathbf{H}_{\mathbb{C}}^2$ and $p, q \in \partial\mathbf{H}_{\mathbb{C}}^2 \setminus \{x\}$. Assume that $p \neq q$ and let $L_{p,q}$ be the projective line passing through p and q . The following formula holds:*

$$d(\pi_x p, \pi_x q) = \frac{d(x, L_{p,q})d(p, q)}{d(x, p)d(x, q)}.$$

Remark 2. *In particular, if K is any compact subset of $\partial\mathbf{H}_{\mathbb{C}}^2$ such that $x \notin K$, the restriction of π_x to K is Lipschitz, and the Lipschitz constant is less than*

$$\frac{1}{d(x, K)^2}.$$

Proof. Let $x = [u]$, $p = [v]$ and $q = [w]$. We are going to show that

$$\frac{\|(u \wedge v) \wedge (u \wedge w)\|}{\|u \wedge v\| \|u \wedge w\|} = \frac{\|u \wedge v \wedge w\| \|v \wedge w\| \|u\| \|v\| \|u\| \|w\|}{\|u\| \|v \wedge w\| \|v\| \|w\| \|u \wedge v\| \|u \wedge w\|}.$$

By Definition 1 and Lemmas 1 and 3, this is equivalent to the Theorem.

After carrying out obvious simplifications, and taking the square of both sides, our goal is to show that

$$\|(u \wedge v) \wedge (u \wedge w)\|^2 = \|u\|^2 \|u \wedge v \wedge w\|^2.$$

By definition, the left-hand side is equal to

$$\begin{aligned} (u \wedge v) \cdot (u \wedge w) &= \begin{vmatrix} (u \wedge v) \cdot (u \wedge v) & (u \wedge v) \cdot (u \wedge w) \\ (u \wedge w) \cdot (u \wedge v) & (u \wedge w) \cdot (u \wedge w) \end{vmatrix} \\ &= \begin{vmatrix} |u \cdot u & u \cdot v| & |u \cdot u & u \cdot w| \\ |v \cdot u & v \cdot v| & |v \cdot u & v \cdot w| \\ |u \cdot u & u \cdot v| & |u \cdot u & u \cdot w| \\ |w \cdot u & w \cdot v| & |w \cdot u & w \cdot w| \end{vmatrix} \end{aligned}$$

Also by definition,

$$\|u\|^2 \|u \wedge v \wedge w\|^3 = u \cdot u \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

Apply Lemma 6 (after interchanging the first two columns of the matrix) to conclude. \square

Lemma 6. *Let $A = (a_{i,j})_{1 \leq i,j \leq 3}$ be a 3×3 matrix. Let $A' = (a'_{i,j})_{1 \leq i,j \leq 2}$ be the 2×2 matrix whose coefficients are the minors*

$$a'_{i,j} = \begin{vmatrix} a_{1,j} & a_{1,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix}.$$

Then

$$\det((A_{i,j})_{1 \leq i,j \leq 2}) = a_{1,2} \times \det((a_{i,j})_{1 \leq i,j \leq 3})$$

Proof. Straightforward. There is a generalization of this formula in Bourbaki, Algebra, III, §8, exercise 2. (To solve the exercise, assume without loss of generality that $a_{1,1} = a_{1,n} = 0$ and $a_{1,2} = a_{1,3} = \dots = a_{1,n-1} = 1$; the argument is then easy.) \square

3 Marstrand's Theorems

3.1 Almost everywhere on the boundary

Proposition 1. *Let A be a Borel subset of $\partial\mathbf{H}_{\mathbb{C}}^2$. For any $x \in \partial\mathbf{H}_{\mathbb{C}}^2$,*

$$\dim_{\mathbb{H}}(\pi_x(A \setminus \{x\})) \leq \dim_{\mathbb{H}}(A).$$

Proof. We may assume (to simplify notations) that x does not belong to A as this does not change the dimension of A . Fix $x \in \partial\mathbf{H}_{\mathbb{C}}^2$ and let K_n a sequence of compact subsets of $\partial\mathbf{H}_{\mathbb{C}}^2 \setminus \{x\}$ such that

$$\partial\mathbf{H}_{\mathbb{C}}^2 \setminus \{x\} = \bigcup_n K_n.$$

For every n , the restriction of π_x to K_n is Lipschitz, thus

$$\dim_{\mathbb{H}}(\pi_x(K_n \cap A)) \leq \dim_{\mathbb{H}}(K_n \cap A) \leq \dim_{\mathbb{H}}(A).$$

Also, $\pi_x(A) = \bigcup_n \pi_x(K_n \cap A)$ and the Proposition follows, since the Hausdorff dimension of a countable union is the supremum of the Hausdorff dimensions. \square

Once Theorem 1 is proved, the arguments for the analogues of Marstrand's Theorems are the usual ones. See [9].

Proposition 2. *Let A be a Borel subset of $\partial\mathbf{H}_{\mathbb{C}}^2$ and let $\alpha = \dim_{\mathbb{H}}(A)$. Assume that $\alpha \leq 2$.*

Then for almost every $x \in \partial\mathbf{H}_{\mathbb{C}}^2$,

$$\dim_{\mathbb{H}}(\pi_x(A \setminus \{x\})) = \alpha.$$

Proof. Denote by ν the Lebesgue measure on $\partial\mathbf{H}_{\mathbb{C}}^2 \simeq S^3$. Let $s < \alpha$. We can find two disjoint compact subsets $Q_0, Q_1 \subset \partial\mathbf{H}_{\mathbb{C}}^2$ such that

- $\nu(Q_0) > 0$;
- Q_1 carries a Borel measure μ such that $I_s(\mu) < \infty$.

Now we compute, using Fubini's theorem,

$$\int d\nu(x) I_s(\pi_x \mu) = \int \frac{d\mu(p)d\mu(q)}{d(p,q)^s} \int d\nu(x) \phi_x(p,q)^{-s}$$

where

$$\phi_x(p,q) = \frac{d(\pi_x p, \pi_x q)}{d(p,q)}.$$

We know (Theorem 1) that

$$\phi_x(p,q) \asymp_{Q_0, Q_1} d(x, L_{p,q})$$

for any $x \in Q_0$ and $p, q \in Q_1$. Also, for any $r > 0$,

$$\nu\{x \in \partial\mathbf{H}_{\mathbb{C}}^2 ; d(x, L_{p,q}) < r\} \lesssim r^2$$

since $L_{p,q}$ is a smooth curve on $\partial\mathbf{H}_{\mathbb{C}}^2 \simeq S^3$. It follows, by a standard argument, that

$$\int d\nu(x) \phi_x(p,q)^{-s}$$

is bounded by a finite constant depending on Q_0, Q_1 and s (recall that $s < 2$).

All in all, $I_s(\pi_x \mu)$ is finite for almost every $x \in Q_0$. This implies that $\dim_{\mathbb{H}}(\pi_x A) \geq s$ for almost every $x \in Q_0$. By covering $\partial \mathbf{H}_{\mathbb{C}}^2$ with adequate pair of compacts subsets, we see that this holds for almost every $x \in \partial \mathbf{H}_{\mathbb{C}}^2$. This being true for any $s < \alpha$, we conclude that $\dim_{\mathbb{H}}(\pi_x A) \geq \alpha$ for almost every x ; the converse inequality is also true by the previous Proposition. \square

Proposition 3 (see [9] Theorem 9.7). *Let μ be a Borel measure on $\partial \mathbf{H}_{\mathbb{C}}^2$ and assume that*

$$I_2(\mu) = \int \frac{d\mu(p)d\mu(q)}{d(p,q)^2} < \infty.$$

Then for almost every $x \in \partial \mathbf{H}_{\mathbb{C}}^2$, the push-forward $\pi_x \mu$ is absolutely continuous.

Let μ be as in the previous proposition. For every $x \in \partial \mathbf{H}_{\mathbb{C}}^2$ such that $\pi_x \mu$ is absolutely continuous, we disintegrate μ above the Lebesgue measure ν_{x^\perp} on $\mathbf{P}(x^\perp)$ (this is the 2-dimensional Hausdorff measure on x^\perp):

$$\mu = \int d\nu_{x^\perp}(y) \mu_{x^\perp, y}.$$

By definition, $\mu_{x^\perp, y}$ is supported on the chain $\pi_x^{-1}(y)$ for almost every $y \in \mathbf{P}(x^\perp)$.

Proposition 4 (see [9] Theorem 10.7). *Let μ be a Borel measure on $\partial \mathbf{H}_{\mathbb{C}}^2$ and assume that $I_s(\mu) < \infty$ with $s > 2$. Then for almost every x and almost every $y \in \mathbf{P}(x^\perp)$,*

$$I_{s-2}(\mu_{x^\perp, y}) < \infty.$$

In particular, $\underline{\dim}(\mu_{x^\perp, y}) = s - 2$.

3.2 Almost everywhere along a fixed chain

Lemma 7. *Let L_1 be a fixed chain and K some compact subset of $\mathbf{P}_{\mathbb{C}}^2 \setminus \overline{\mathbf{H}_{\mathbb{C}}^2}$. Then for any $r > 0$ small enough and any chain $L_2 = \mathbf{P}(w^\perp) \cap \partial \mathbf{H}_{\mathbb{C}}^2$ where $[w] \in K$,*

$$\text{diam}\{x \in L_1 ; d(x, L_2) \leq r\} \underset{L_1, K}{\lesssim} r.$$

Two distinct chains cannot be tangent to one another; the content of this Lemma is that they cannot be “nearly tangent” if they are not allowed to degenerate. We skip the proof which is a straightforward computation.

Let L_1 and K be as in the Lemma and denote by ν the Lebesgue measure on L_1 . For any chain L_2 as in the Lemma and for r small enough, we get

$$\nu\{x \in L_1 ; d(x, L_2) \leq r\} \underset{L_1, K}{\lesssim} r.$$

We deduce the following results.

Proposition 5. *Fix some chain L . Let A be a Borel subset of $\partial \mathbf{H}_{\mathbb{C}}^2$ that does not meet L and let $\alpha = \dim_{\mathbb{H}}(A)$.*

Then for almost every $x \in L$,

$$\dim_{\mathbb{H}}(\pi_x(A)) \geq \inf\{\alpha, 1\}.$$

This bound is unlikely to be sharp when $\alpha > 1$.

Proposition 6. *Fix some chain L . Let A be some Borel subset of $\partial\mathbf{H}_{\mathbb{C}}^2$ that does not meet L . For any $s < \inf\{\dim_{\mathbb{H}}(A), 1\}$,*

$$\dim_{\mathbb{H}}\{x \in L ; \dim_{\mathbb{H}}(\pi_x A) < s\} \leq s.$$

Proposition 5 is an example of a Marstrand’s projection Theorem for a family of projections (onto planes) parametrized by a curve in the 3-dimensional space. The parameter space has dimension 1, whereas the Grassmannian has dimension 3.

Some authors have studied this kind of problem for *orthogonal* projections, see *e.g.* [7] and [10].

It seems likely that for sets of dimension > 1 , Proposition 5 could be improved. We thus ask

Question. *For $\alpha > 1$, what is the largest $\sigma(\alpha) > 1$ such that, with the notations of Proposition 5,*

$$\dim_{\mathbb{H}}(\pi_x A) \geq \sigma(\alpha)$$

for almost every x ?

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